

# Existence of positive equilibria for quasilinear models of structured population

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## Abstract

In this paper I prove the existence of a positive stationary solution for a generic quasilinear model of structured population. The existence is proved using Schauder's fixed point theorem. The theorem is applied to a hierarchically size-structured population model.

**Keywords:** structured population model, stationary solution, net reproduction function, compactness, Schauder's fixed point theorem.

## 1 Introduction

The size-structured population model IBVP (Initial Boundary Value Problem, see [3]),

in the autonomous case, has the following general form:

$$\begin{cases} u_t + (g(x, u(t, \cdot)) u)_x + \mu(x, u(t, \cdot)) u = 0 \\ g(0, u(t, \cdot)) u(t, 0) = \int_J \beta(x, u(t, \cdot)) u(t, x) dx, \end{cases} \quad (1)$$

where  $x \in J = [0, \infty)$  represents *age* or *size*,  $t \geq 0$  is *time*,  $u$  is the *population density*,  $u(t, \cdot) \in L^1(J)$  for each  $t \geq 0$ .

The model equations involve the following vital rates:  $\mu = \mu(x, u)$  — mortality,  $\beta = \beta(x, u)$  — fertility and  $g = g(x, u)$  — growth rate. These coefficients depend on the size  $x$  and on the total population behaviour through  $u$  in a general (also nonlinear) way.

The total population at the instant  $t$  is given by  $P(t) = \int_J u(t, x) dx$ , the flow of the newborns is  $B(t) = \int_0^\infty \beta(x, u(t, \cdot)) u(t, x) dx$ . In this paper we obtain for Pbm. (1) a theorem of existence of a positive equilibrium.

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In general, however, the well-posedness of this class of PDE models is still an open question ([4], Introduction).

The first nonlinear population model was introduced and analysed in the seminal paper [8] of Gurtin and MacCamy in 1974, with nonlinearities depending only on  $P(t)$ . It was followed in the eighties by several other papers with generic nonlinearities in  $u$  for the case  $g = 1$  e.g. by J. Prüss that gave some sufficient conditions for the existence of a positive equilibrium [9, 10, 11].

In 2003 Diekmann et al. [5] managed the case of nonconstant  $g$  and  $n$  scalar biomasses  $S_1, S_2, \dots, S_n$  depending on  $u$ , using a very different mathematical formulation; they proved the existence of nonzero equilibria and gave bifurcation conditions.

In 2006 Farkas e Hagen [7] studied the stability of stationary solutions of the IBVP, in the case of nonlinear dependence on the total population  $P$ , via linearization and semigroup and spectral methods. They give stability criteria in terms of a modified net reproduction rate.

In this paper I establish Thm. 5, that gives sufficient conditions for the existence of a positive equilibrium for Pbm. (1), under generic dependence on  $u$ . I use a compactness hypothesis. I set also preliminarily some positivity and boundedness hypotheses on the coefficients  $\mu$  and  $g$ .

The problem is transformed in a fixed point problem and the existence of a solution is obtained through Schauder's fixed point theorem.

However there is no uniqueness in general. I give a made-up counterexample. I give also a condition for the non-existence of positive equilibria using suitable assumptions of monotonicity on the coefficients  $\mu$ ,  $g$  and  $\beta$ .

At the end of Sec. 3, I show as application the existence of a positive stationary solution for a nonlinear model of structured population of Ackleh and Ito [2].

In the Appendix, I resume some propositions on compactness.

## 2 Preliminaries

### 2.1 Notations

$J = [0, \infty)$  is the interval of definition of  $x$ .

$\langle g, f \rangle = \int_J g(x) f(x) dx$  for  $f \in L^1(J)$  and  $g \in L^\infty(J)$ .

$L_+^1(J) = \{\phi \in L^1(J) \mid \phi(x) \geq 0 \text{ a.e. } x \in J\}$  is the positive cone of  $L^1(J)$ .

Given two functions  $u_1, u_2: [0, \infty) \rightarrow [0, \infty)$ , we will write  $u_1 < u_2$  if  $0 \leq u_1(x) \leq u_2(x)$  and  $u_1 \neq u_2$  a. e.  $x \in J$ . The relation  $<$  is a partial order on the cone  $L_+^1(J)$ .

If  $e_1, e_2 \in L^1(J)$  and  $e_1 < e_2$ , then write

$$[e_1, e_2] = \{\phi \in L^1(J) \mid e_1(x) \leq \phi(x) \leq e_2(x) \text{ a.e. } x \in J\}.$$

Functions  $f(u(\cdot))$  defined for  $u \in L_+^1(J)$  will be usually briefly denoted as  $f(u)$ .

## 2.2 Hypotheses and definitions

### Hypothesis (A)

- a) The functions  $x \mapsto g(x, u)$ ,  $\mu(x, u)$  are  $L^\infty(J)$  for each  $u \in L_+^1(J)$  and there exist constants  $\underline{g}, \overline{g}, \underline{\mu}, \overline{\mu}$ :

$$0 < \underline{g} \leq g(x, u) \leq \overline{g}, \quad 0 < \underline{\mu} \leq \mu(x, u) \leq \overline{\mu}$$

for each  $u \in L_+^1(J)$ , a. e.  $x \in J$ .

- b)  $\beta(x, u) \geq 0$  for a.e.  $x \in J$ , for each  $u \geq 0$  and there exists a constant  $\overline{\beta} > 0$ :  $\beta(x, u) \leq \overline{\beta}$  for each  $u \geq 0$ , a. e.  $x \in J$ .
- c)  $u \mapsto g(x, u), \mu(x, u), \beta(x, u)$  are continuously depending on  $u \in L_+^1(J)$  for a.e.  $x \in J$ .

**Auxiliary functions.** For  $x \in J$ ,  $u \in L_+^1(J)$ , we set:

$$\Pi(x, u) := \frac{1}{g(x, u)} e^{-\int_0^x \frac{\mu(y, u)}{g(y, u)} dy}. \quad (2)$$

Under the boundedness assumptions of Hyp. (A), we define the auxiliary functions  $e_1, e_2$ :

$$e_1(x) := \frac{e^{-(\overline{\mu}/\underline{g})x}}{\underline{g}}, \quad e_2(x) := \frac{e^{-(\underline{\mu}/\overline{g})x}}{\underline{g}}. \quad (3)$$

**Lemma 1 (Properties of  $\Pi$ )** *Using the assumptions on the lower and upper bounds of  $\mu$  and  $g$ , given in Hyp. (A), we obtain for each  $x, u$ :*

$$e_1(x) \leq \Pi(x, u) \leq e_2(x). \quad (4)$$

Moreover  $\Pi(\cdot, u) \in L^1(J) \cap L^\infty(J)$  for each  $u \in L_+^1(J)$ .

The interval  $[e_1, e_2]$  is a closed convex subset of  $L_+^1(J)$ .

**“Onion” set.** Set  $U := \bigcup_{\lambda > 0} [\lambda e_1, \lambda e_2]$ .

It is simple to prove that the sets  $U$  and  $\overline{U} = U \cup \{0\}$  are convex.

**Hypothesis (C)** (*Uniformly bounded variation*).

$$\forall T > 0 : \quad \lim_{h \rightarrow 0} \sup_{u \in U} \int_0^T |g(x+h, u) - g(x, u)| dx = 0.$$

We mean that  $g$  is extended as 0 for  $x < 0$ .

**Remark 1** Condition (C) means that sup has to be considered on functions of the form  $u = \lambda v$ , with  $v \in [e_1, e_2]$ . Since  $U \neq L_+^1(J)$  (e.g.  $x^{-1/2}e^{-x} \notin U$ ) this is an effective reduction of the requests.

Under Hyp. (A), Condition (C) is satisfied also for  $u = 0$  (therefore it holds for  $u \in \overline{U}$ ) because  $g(x, u)$  is continuous in  $u$ .

### Hypothesis (D)

$\forall T > 0 : \exists k_T > 0 : |g_x(x, u)| \leq k_T$ , for each  $u \in U$  and a.e.  $x \in [0, T]$ .

Hyp. (D) implies Hyp. (C).

**Hypothesis ( $L_\beta$ ) (*Limit of  $\beta$* ).** For each  $x \geq 0$ ,

$$\lim_{\|u\|_1 \rightarrow +\infty, u > 0} \beta(x, u) = 0.$$

**Definition 2 (*Net reproduction function*)** (Cmp. ([10], p. 330) For  $u \in L_+^1(J)$

$$R(u) := \int_J \beta(x, u) \Pi(x, u) dx. \quad (5)$$

Under Hyp. (A),  $R(u)$  is well-defined and if also ( $L_\beta$ ) holds, then

$$\lim_{\|u\|_1 \rightarrow \infty, u \geq 0} R(u) = 0. \quad (6)$$

## 2.3 Compactness

The (closed, convex) interval  $[e_1, e_2] \subseteq [0, e_2] \subseteq L_+^1(J)$  is invariant with respect to  $\Pi$ , i.e.  $\Pi(\cdot, [e_1, e_2]) \subseteq \Pi(\cdot, [0, e_2]) \subseteq [e_1, e_2]$ .

**Lemma 3 (Compactness)** Under Hyp. (A) and (C), the function  $u \mapsto \Pi(\cdot, u)$ , defined on  $U$  and  $\bar{U}$  in  $L^1(J)$ , is compact.

The lemma of compactness is proved in Appendix, Sec. B.

## 3 Existence of equilibria

In this section we prove the existence of a positive stationary solution  $u^*$  for Pbm. (1) as fixed point of a suitable transformation of  $L_+^1(J)$ .

### 3.1 Stationary solutions

The equilibria are the time-independent solutions  $u = u^*(x)$  of Problem (1). These are determined from

$$\begin{cases} \frac{\partial}{\partial x} (g(x, u^*(\cdot)) u^*(x)) + \mu(x, u^*(\cdot)) u^*(x) = 0 \\ g(0, u^*(\cdot)) u^*(0) = \int_0^\infty \beta(x, u^*(\cdot)) u^*(x) dx \end{cases} \quad (7)$$

and (see [10], Eq. (8)) they corresponds to the solutions of the functional equation

$$u(x) = \frac{\int_0^\infty \beta(x', u) u(x') dx'}{g(x, u)} e^{-\int_0^x \frac{\mu(y, u)}{g(y, u)} dy} \quad \text{for } x \in J, \quad (8)$$

the only premises being  $g > 0$ ,  $\frac{\mu(\cdot, u)}{g(\cdot, u)} \in L_{loc}^1(J)$ .

This equation is translated immediately in a fixed point problem.

**Proposition 4** Under Hyp. (A) the stationary solutions of Pbm. (1) are the fixed points of the functional  $\mathcal{T}: L_+^1(J) \rightarrow L_+^1(J)$  defined as

$$(\mathcal{T}\phi)(x) = \frac{G(\phi)}{g(x, \phi)} e^{-\int_0^x \frac{\mu(y, \phi(\cdot))}{g(y, \phi(\cdot))} dy} \quad (9)$$

and vice versa, where  $G: L_+^1(J) \rightarrow \mathbb{R}$  is given by

$$G(\phi) = \int_J \beta(x, \phi(\cdot)) \phi(x) dx,$$

for  $\phi \in L_+^1(J)$ .

The functional equation  $u = \mathcal{T}u$  can be written in a more compact form as

$$u(x) = G(u(\cdot)) \Pi(x, u(\cdot)), \quad (10)$$

that we discuss.

**Theorem 5 (Existence of equilibria)** Assume Hyp. (A) and (C). Suppose there is a constant  $\rho_0 > 0$  such that for  $u \in L_+^1(J)$ ,  $\|u\|_1 \geq \rho_0$  implies  $R(u) \leq 1$ . If  $R(0) > 1$  then Problem (1) admits at least a positive stationary solution.

The solution satisfies the functional equation

$$u^*(x) = \lambda^* \Pi(x, u^*(\cdot)), \quad (11)$$

where  $\lambda^* > 0$  is a suitable number and the corresponding population is constant and given by  $P^* = \lambda^* \|\Pi(\cdot, u^*)\|_1$ .

From (6) we have the following statement:

**Corollary 6** Under Hyp. (A), (C) and  $(L_\beta)$ , if  $R(0) > 1$  then Problem (1) admits a positive stationary solution.

### 3.2 Proof of Thm. 5

Prop. 4 reduces the search for equilibria of Pbm. (1) to Eq. (10).

$G(0) = 0$  gives the trivial equilibrium  $u = 0$  so we exclude this case.

The proof is divided into two steps.

(i) **Splitting variables.** Consider Eq. (10): assume that  $u$  is a solution of  $u = G(u) \Pi(\cdot, u)$ .

Set  $\lambda := G(u)$  ( $\neq 0$ ) and  $v = \frac{1}{\lambda} u$ .

By substitution we obtain:  $\lambda v = \lambda \Pi(x, \lambda v)$  and  $\lambda = \int_0^\infty \beta(x, \lambda v) \lambda \Pi(x, \lambda v) dx$  so that  $1 = \int_0^\infty \beta(x, \lambda v) \Pi(x, \lambda v) dx$ . Hence  $(v, \lambda) \in [e_1, e_2] \times (0, \infty)$  is a solution of the system:

$$\begin{cases} v(x) = \Pi(x, \lambda v(\cdot)), \\ R(\lambda v) = 1. \end{cases} \quad (12)$$

Vice versa, if  $(v, \lambda)$  is a solution of (12), then  $u = \lambda v$  is a solution of the equation  $u = G(u) \Pi(\cdot, u)$ .

The condition  $R(0) > 1$  implies that  $\lambda^* \neq 0$ . For each solution  $(v, \lambda)$  of Pbm. (12) we have  $(v, \lambda) \in [e_1, e_2] \times (0, \infty)$ .

(ii) **Fixed point.** In this step we apply Schauder's fixed point theorem — see [6], [12]. We write Pbm. (12) in the form

$$\begin{cases} v(\cdot) = \Pi(\cdot, \lambda v), & v \in [e_1, e_2], \\ \lambda = \max\{\lambda + R(\lambda v) - 1; 0\}, & \lambda \geq 0 \end{cases} \quad (13)$$

that is  $(v, \lambda) = A((v, \lambda))$ , with  $(v, \lambda) \in [e_1, e_2] \times (0, \infty)$  and  $A$  defined by the second members of (13).

The map  $u \mapsto \Pi(\cdot, u)$  is continuous and compact on  $U$ ; the function  $R(u)$  is continuous and bounded from  $L_+^1(J)$  to  $(0, \infty)$ , since  $0 < R(u) \leq \bar{\beta} \|e_2\|_1$ ; therefore  $A: [e_1, e_2] \times (0, \infty) \rightarrow L^1(J) \times (0, \infty)$  is continuous and compact.

$A_1(v, \lambda) := \Pi(\cdot, \lambda v)$  has image in  $[e_1, e_2]$ .

Now prove that for a fixed  $M > \frac{\rho_0}{\|e_1\|_1}, \frac{\rho_0}{\|e_1\|_1} + \bar{\beta} \|e_2\|_1 - 1$ , then

$A_2(v, \lambda) := \max\{\lambda + R(\lambda v) - 1; 0\}$  maps  $[e_1, e_2] \times [0, M]$  on  $[0, M]$ .

If  $\frac{\rho_0}{\|e_1\|_1} \leq \lambda \leq M$ , then  $\lambda \geq \frac{\rho_0}{\|v\|_1}$  and  $R(\lambda v) \leq 1$ , so that

$$\lambda + R(\lambda v) - 1 \leq 1 + \lambda - 1 = \lambda \leq M.$$

If  $0 \leq \lambda < \rho_0/\|e_1\|_1$ , then  $\lambda + R(\lambda v) - 1 \leq \frac{\rho_0}{\|e_1\|_1} + \bar{\beta} \|e_2\|_1 - 1 \leq M$ .

So  $A$  maps  $[e_1, e_2] \times [0, M]$ , a closed convex subset of  $L^1(J) \times (0, \infty)$ , in itself.

Since  $A$  is compact, by Schauder's fixed point theorem, Eq. (13) has at least a fixed point  $(v^*, \lambda^*) \in [e_1, e_2] \times [0, M]$  and it is different from 0 for the initial remark;  $(v^*, \lambda^*)$  is a fixed point also for Eq. (12).

Finally, Eq. (10) is satisfied by  $u^* = \lambda^* v^*$  and the corresponding stationary population is

$$P^* = \int_J u(x) dx = \lambda^* \int_J v^*(x) dx.$$

**Remark 2**  $R(0) > 1$  implies  $\bar{\beta} \|e_2\|_1 > 1$ , therefore in the proof it is possible to assume  $M = \frac{\rho_0}{\|e_1\|_1} + \bar{\beta} \|e_2\|_1 - 1$  and to have the estimate  $P^* \leq M \|e_2\|_1$ .

### 3.3 A counterexample

Thm. 5 is a sufficient condition but not a necessary one. We can have also  $R(0) < 1$  if there exists  $u_0 \in L_+^1(J)$  such that  $R(u_0) > 1$ . In this case it is possible to need other conditions on  $u_0$  to prove a statement of existence. The idea is to construct explicitly an example with a positive equilibrium but  $R(0) < 1$ .

Set  $\mu(x, u) = g(x, u) = g$  so that  $\Pi(x, u) = \frac{1}{g} e^{-x}$ , independent of  $u$ .

Define  $e_0(x) := e^{-x}$ . Take  $F: L_+^1(J) \rightarrow \mathbb{R}, u \mapsto F(u)$ , such that  $F(0) < 1$ ,  $F(e_0) = 1$  and  $\lim_{\|u\|_1 \rightarrow \infty, u > 0} F(u) = 0$ ,  $F$  continuous but obviously nonmonotonic.

Now set  $\beta(x, u) = 2g(1 - e^{-x})F(u)$  so that  $R(u) = F(u)$ .

Then  $R(e_0) = 1$  and  $u = e_0$  is a solution of the fixed point equation and a positive equilibrium.

As example of function  $F$  we can take  $F(u) := f(\|u\|_1)$ . where

$$f(a) := \begin{cases} \frac{1}{2} + 3a & \text{for } 0 \leq a \leq \frac{1}{2}, \\ 3 - 2a & \text{for } \frac{1}{2} < a \leq \frac{5}{4}, \\ \frac{e^{5/4}}{2} e^{-a} & \text{for } a > \frac{5}{4}. \end{cases} \quad (14)$$

In this case we have *two* positive equilibria,  $u(x) = e^{-x}$  and  $u(x) = \frac{1}{6}e^{-x}$ , corresponding to the two solutions of  $f(a) = 1$ , i. e.  $a = 1$ ,  $a = 1/6$ .

### 3.4 A nonexistence result and a sufficient and necessary condition

Under suitable monotonicity hypotheses,  $R(0) > 1$  becomes a necessary and sufficient condition.

A function  $f$ , defined on ordered spaces, is *increasing* if  $u_1 < u_2$  implies  $f(u_1) < f(u_2)$ . The other monotonicity definitions are extended in the same ways.

Now assume  $u \in L_+^1(J)$  in the following statements.

#### Assumption (M) (*Monotonicity*)

- $u \mapsto \mu(x, u)/g(x, u)$  is nondecreasing (or increasing) for each  $x \geq 0$  (*mortality-growth ratio*),
- $u \mapsto \beta(x, u)/\mu(x, u)$  is decreasing (or nonincreasing) for each  $x \geq 0$  (*fertility-mortality ratio*),
- $x \mapsto \beta(x, u)/\mu(x, u)$  is nondecreasing (or increasing) for each  $u$ .

The hypotheses between parentheses are in alternative:  $u \mapsto \beta/\mu$  must be strictly decreasing and the other two functions are only nondecreasing, or, vice versa,  $u \mapsto \beta/\mu$  nonincreasing and the others have to be two strictly increasing.

To prove the nonexistence condition we need the following statement:

**Lemma 7 (Monotonicity)** *Assume Hypotheses (A), (C),  $(L_\beta)$  and Assumption (M).*

*Then the functional  $R: L_+^1(J) \rightarrow (0, \infty)$  is continuous, decreasing and*

$$\lim_{\|u\|_1 \rightarrow +\infty, u > 0} R(u) = 0.$$

I do not give the details of the proof of this lemma, but the main idea is to write  $R(u)$  as  $\int_J dx \frac{\beta(x,u)}{\mu(x,u)} \frac{\mu(x,u)}{g(x,u)} e^{-\int_0^x \frac{\mu(y,u)}{g(y,u)} dy}$  and to study the properties of monotonicity of the integral  $\int_J dx h(x) f(x) e^{-\int_0^x f(y) dy}$  with respect to suitable  $f$  and  $h$ .

For a detailed proof, see Bertoni [1].

**Proposition 8 (Non existence of positive stationary solutions)**

*Under Hypotheses of Lemma 7, if  $R(0) \leq 1$  then Pbm. (1) has no positive stationary solutions.*

*Proof.* If  $R(0) \leq 1$  then  $R(u) = 1$  does not have positive solutions by monotonicity.

Since existence of positive equilibria is equivalent to positive solutions of  $u = G(u) \Pi(\cdot, u)$  and so of Eq. (12), the conclusion follows.

As consequence, Condition  $R(0) > 1$  becomes a *necessary and sufficient condition of existence of positive equilibria* for Pbm. (1) under Hyp. (A), (C),  $(L_\beta)$  and (M).

### 3.5 Applications

Ackleh e Ito [2] consider a hierarchically size-structured population model that can be reported to Eq. (1). They proved existence of measure-valued solutions for the Cauchy problem. We give a condition of existence of a stationary positive solution for a simple case of this model, by taking

$$g(x, u(\cdot)) = \underline{g} + (\bar{g} - \underline{g}) e^{-\int_x^\infty u(y) dy}. \quad (15)$$

Hyp. (D) is equivalent to

$$\forall T > 0 : \sup_{u \in U} \operatorname{ess\,sup}_{0 \leq x \leq T} |g_x(x, u)| < \infty$$

that is, for (15):

$$\forall T > 0 : \sup_{u \in U} \operatorname{ess\,sup}_{0 \leq x \leq T} |e^{-\int_x^\infty u(y) dy} \cdot u(x)|_\infty < \infty. \quad (16)$$

For  $u = \lambda v$  with  $v \in [e_1, e_2]$  we use the inequality  $\sup_{\lambda > 0} \lambda e^{-\alpha \lambda} = \frac{1}{\alpha e}$ : therefore

$$e^{-\int_x^\infty u(y) dy} \cdot u(x) = \lambda v(x) e^{-\lambda \int_x^\infty v(y) dy} \leq \frac{v(x)}{e \int_x^\infty v(y) dy} \leq \frac{e_2(x)}{e \int_T^\infty e_1(y) dy} < \infty.$$

Assume  $\mu$  and  $\beta$  to satisfy Hyp. (A) and  $(L_\beta)$ . The other conditions on  $g$  of Cor. 6 are trivially satisfied, so we obtain the existence of at least one positive stationary solution if

$$\int_J dx \beta(x, 0) e^{-\int_0^x \frac{\mu(y, 0)}{g(y, 0)} dy} > \bar{g}.$$



## Appendix

### A Compactness conditions

As well known, the conditions for the relative compactness of a set  $W$  in  $L^1(0, \infty)$  are given by the Riesz–Kolmogorov Theorem. We use the following version:

i)  $W$  is bounded;

$$ii) \lim_{T \rightarrow \infty} \sup_{w \in W} \int_{x > T} |w(x)| dx = 0.$$

$$iii) \lim_{h \rightarrow 0} \sup_{u \in W} \int_0^T |w(x+h) - w(x)| dx = 0 \text{ for each } T > 0.$$

Sets of continuous, uniformly bounded variation functions in  $L^1(0, \infty)$  are (relatively) compact.

### B Compactness of $\Pi$ (Proof of Lemma 3)

For each  $u \in L_+^1(J)$ , the function  $\Pi(x, u)$  defined by (2) has the following properties:

1.  $\Pi(\cdot, u) \in [e_1, e_2]$ , that implies (i) and (ii) of the Riesz–Kolmogorov Theorem;
2.  $x \mapsto \Pi(x, u)$  is continuous.

Now we prove (iii) for  $u \in \overline{U}$ . Let be  $T > 0$ ,  $h > 0$ :

$$\begin{aligned} & \int_0^T |\Pi(x+h, u) - \Pi(x, u)| dx \leq \\ & \leq \int_0^T dx \frac{e^{-\int_0^{x+h} \frac{\mu(y, u)}{g(y, u)} dy}}{g(x+h, u)} \left( e^{\int_x^{x+h} \frac{\mu(y, u)}{g(y, u)} dy} - 1 \right) + \\ & + \int_0^T dx \frac{e^{-\int_0^x \frac{\mu(y, u)}{g(y, u)} dy}}{g(x, u) g(x+h, u)} |g(x+h, u) - g(x, u)| \leq \\ & \leq \frac{T \overline{\mu}}{\underline{g}^2} h + \frac{T}{\underline{g}^2} \int_0^T dx |g(x+h, u) - g(x, u)|, \end{aligned}$$

therefore, using Hyp. (C) for  $u \in \overline{U}$ , this completes the proof. The case  $h < 0$  is managed analogously.

We obtain the  $\Pi$  sends  $U$  in a relatively compact subset of  $[e_1, e_2]$  in the norm of  $L^1(J)$  i. e. the set  $\Pi(\cdot, U)$  is relatively compact.

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